# Factorization of Feynman graphs at finite temperature and chemical potential

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# Overview

- Perturbative calculations are still needed and/or useful in XXI century QFT at finite $T$ and $\mu$.

- Remarkable and amusing general property of Feynman *loop* graphs in thermal field theories, undiscovered until recently:
  *"Thermal part" can be simply related to "vacuum part"*.

- Result is valid for a general field theory involving scalar, fermionic and gauge fields.

- Result holds in both Euclidean Time and Real Time (closed time path) formalisms.

- Result can be extended to non-zero chemical potential.

- Warning: this is a talk about the formalism of thermal QFT
  No new physics here! (sorry)

# Published work

- C.Dib, O.E. and I.Schmidt, hep-ph/**9704**078, PLB (1997)
  3-dimensional Rules for Finite-Temperature Loops
- O.E. and E. Stockmeyer, hep-ph/**0305**001, PRD (2004)
  An operator representation for Matsubara sums
- O.E., hep-ph/**0501**273, PRD (2005)
  The thermal operator representation for Matsubara sums
- F.Brandt, A.Das, O.E., J.Frenkel and S.Perez, hep-th/**0508**067, PRD (2005)
  Thermal Operator Representation of Finite Temperature Graphs
- F.Brandt, A.Das, O.E., J.Frenkel and S.Perez, hep-th/**0601**224, PRD (2006)
  Thermal Operator Representation of Finite Temperature Graphs. II.
- F.Brandt, A.Das, O.E., J.Frenkel and S.Perez, hep-th/**0601**227, PRD (2006)
  Factorization of finite temperature graphs in thermal QED

# (Some) previous related work...

- M. Gaudin (Nuovo Cimento, 1965)
  *In French*

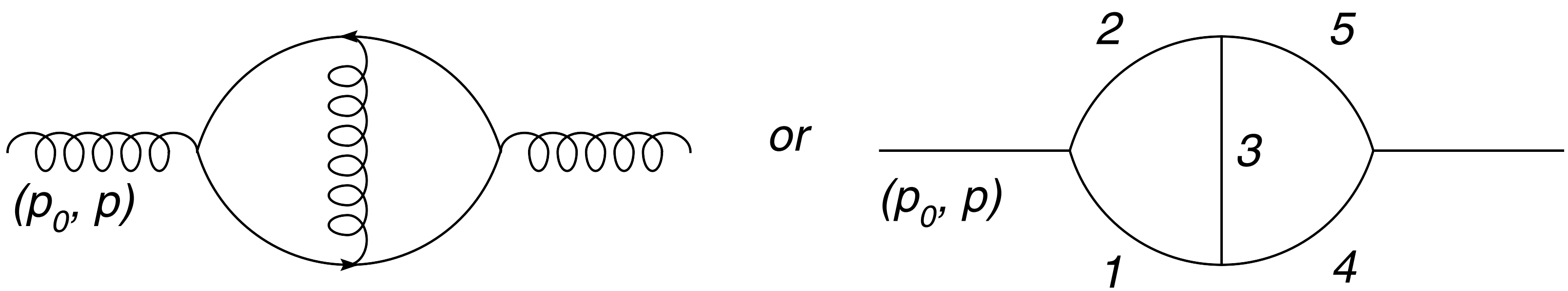
- R. Pisarski (NPB, 1988),
  *Computing finite temperature loops with ease*

  **Abstract.** An efficient way of calculating perturbatively at non-zero temperature is to start with a diagram in momentum space, and then Fourier transform each propagator in a loop with respect to the (imaginary) time. Discontinuities are read off from the energy denominators of this non-covariant approach.

- F. Guerin (PRD, 1994),
  *Rules for diagrams in thermal field theories*

  **Abstract.** Sets of rules are proposed that allow one to write down the amplitude associated with a diagram at temperature T once the energy running around each loop has been summed over, in the imaginary-time formalism. Alternative forms are given: one is based on tree diagrams, another one on possible intermediate states. A close analogy to the T=0 case is obtained. The amplitude's analytic structure is explicit. A factorization property is found for the N-point imaginary-time Green functions.

# A sample diagram in QED or $g\phi^3$




$$\text{Diagram} = \int [\text{loop spatial momenta } \mathbf{k}_l]\, \gamma^{(T)}(p_0, E_i)$$

$$\gamma^{(T)}(p_0, E_i) = T^L \sum_{\substack{\text{loop Matsubara} \\ \text{frequencies } \omega_l}} \prod_i \frac{1}{\omega_i^2 + E_i^2}$$

- $i = 1, 2, \ldots, I =$ total number of internal lines
- $E_i = \left(\mathbf{k}_i^2 + m^2\right)^{1/2}$
- $\omega_i =$ linear combination of $p_0$ and $\omega_l$
- $\omega_l = (2\pi T)n_l, \quad n_l =$ integer

# Main result

$$\gamma^{(T)}(p_0, E_i) = \mathcal{O}^{(T)}(E_i)\gamma^{(0)}(p_0, E_i)$$

where

$$\gamma^{(0)}(p_0, E_i) = \int \frac{dk_{0l}}{2\pi} \prod_i \frac{1}{k_{0i}^2 + E_i^2} \qquad T = 0 \text{ energy loop integral}$$

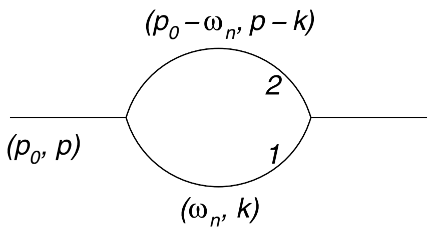
and

$$\mathcal{O}^{(T)}(E_i) = \prod_{i=1}^{I} \left(1 + n_i \left(1 - S_i\right)\right) \qquad \textit{Thermal Operator}$$

defined in terms of

- ▷ $n_i = n_{BE}(E_i) = \left(e^{E_i/T} - 1\right)^{-1}$ Bosonic thermal occupation factor
- ▷ $S_i f(\ldots, E_i, \ldots) = f(\ldots, -E_i, \ldots)$ Reflection operator

# Explicit simple example ($g\phi^3$ theory)




$$E_1 = \left(k^2 + m^2\right)^{1/2}, \quad E_2 = \left((\mathbf{p}-\mathbf{k})^2 + m^2\right)^{1/2}$$

$$\gamma^{(T)}(p_0, E_1, E_2) = \frac{1 + n_1 + n_2}{ip_0 + E_1 + E_2} - \frac{n_1 - n_2}{ip_0 + E_1 - E_2} + \frac{n_1 - n_2}{ip_0 - E_1 + E_2} - \frac{1 + n_1 + n_2}{ip_0 - E_1 - E_2}$$

$$= \mathcal{O}^{(T)}(E_1, E_2)\gamma^{(0)}(p_0, E_1, E_2)$$

where

$$\gamma^{(0)}(p_0, E_1, E_2) = \frac{1}{ip_0 + E_1 + E_2} - \frac{1}{ip_0 - E_1 - E_2}$$

and

$$\begin{aligned} \mathcal{O}^{(T)}(E_1, E_2) &= (1 + n_1 (1 - S_1)) (1 + n_2 (1 - S_2)) \\ &= 1 + n_1 (1 - S_1) + n_2 (1 - S_2) + n_1 n_2 (1 - S_1)(1 - S_2) \end{aligned}$$

# Explicit simple example ($g\phi^3$ theory) (continued)

Note that

$$(1-S_1)(1-S_2)\gamma^{(0)}(p_0,E_1,E_2) = (1-S_1)(1-S_2)\left[\frac{1}{ip_0+E_1+E_2} - \frac{1}{ip_0-E_1-E_2}\right] \equiv 0$$

So,

$$\gamma^{(T)}(p_0,E_1,E_2) = [1+n_1(1-S_1)+n_2(1-S_2)]\gamma^{(0)}(p_0,E_1,E_2)$$

# Generalization

The *Thermal Operator* can also be written as:

$$\mathcal{O}^{(T)}(E_i) := 1 + \sum_{i=1}^{I} n(E_i)(1 - \mathcal{S}_i) + {\sum_{\langle i_1, i_2 \rangle}}' n(E_{i_1}) n(E_{i_2})(1 - \mathcal{S}_{i_1})(1 - \mathcal{S}_{i_2})$$

$$+ \cdots + {\sum_{\langle i_1, \ldots, i_L \rangle}}' \prod_{l=1}^{L} n(E_{i_l})(1 - \mathcal{S}_{i_l}).$$

where

- the indices $i_1, i_2, \ldots$ run from 1 to $I$.
- $\langle i_1, \ldots, i_k \rangle$ represents a given set of $k$ internal lines.
- $\sum'$ means that those tuples $\langle i_1, \ldots, i_k \rangle$ that are *cut sets* of the diagram must be excluded.

For instance, for the diagram




$\mathcal{O}^{(T)}$ has no terms with $n_1 n_2$ or $n_4 n_5$.

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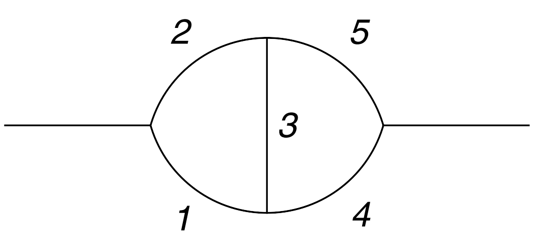


$\mathcal{O}^{(T)}$ has no terms with $n_1 n_2$ or $n_4 n_5$.

# The Thermal Operator

Properties of $\mathcal{O}^{(T)}(E_i)$:

- it is *real* and *linear*
- it is (effectively) of degree $L$ in the $n_i$'s [in the Euclidean formalism]
- it is *independent of the external energies* $p_0 = \{p_{01}, p_{02}, \ldots\}$
- it is different for each diagram
- it is an idempotent operator: $\mathcal{O}^{(T)}(E_i)\mathcal{O}^{(T)}(E_i) = \mathcal{O}^{(T)}(E_i)$

# Proof

- ▷ Use of Gaudin's method to perform Matsubara sums (J.P.Blaizot+Reinosa hep-ph/**0406**109, O.E. hep-ph/**0501**273))

- ▷ Use of mixed time-momentum representation (F.Brandt+A.Das+O.E.+J.Frenkel+S.Perez, hep-th/**0508**067)

## Zero temperature

$$\begin{aligned}\Delta^{(0)}(\tau,E) &= \int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\,e^{-ip_0\tau}\Delta^{(0)}(p_0,E)\\ &= \frac{1}{2E}\left[\theta(\tau)e^{-E\tau}+\theta(-\tau)e^{E\tau}\right], \qquad -\infty<\tau<\infty\end{aligned}$$

## Finite temperature

$$\begin{aligned}\Delta^{(T)}(\tau,E) &= T\sum_{n=-\infty}^{\infty}e^{-ip_0\tau}\Delta^{(0)}(p_0,E)\Big|_{p_0=(2\pi T)n}\\ &= \frac{1}{2E}\left[\theta(\tau)\left\{(1+n(E))\,e^{-E\tau}+n(E)e^{E\tau}\right\}\right.\\ &\qquad \left.+\theta(-\tau)\left\{n(E)e^{-E\tau}+(1+n(E))\,e^{E\tau}\right\}\right], -\frac{1}{T}<\tau<\frac{1}{T}\end{aligned}$$

# Proof (continued)

It holds that

$$\Delta^{(T)}(\tau, E) = [1 + n(E)(1 - S(E)]\, \Delta^{(0)}(\tau, E)$$

$$= O^{(T)}(E)\Delta^{(0)}(\tau, E)$$

*Basic* (bosonic) thermal operator

Then

- $O^{(T)}(E)$ is *independent* of the time variable $\tau$
- Thus, $O^{(T)}(E)$ can be taken out of finite temperature loop integrals (which contain integrations over internal times over the ranges $0 \leq \tau \leq 1/T$)
- It can be shown that the integration ranges can then be extended to $-\infty < \tau < \infty$ (thermal operator annihilates added parts)

# Real-time (closed time path) formalism

Matrix valued propagators

$$\Delta_{ab}^{(T)}(p) = \Delta_{ab}^{(0)}(p) + 2\pi n(|p_0|)\delta\left(p^2 - m^2\right), \qquad a, b = +, -$$

Mixed time-momentum representation

$$\Delta(t, \vec{p}) = \Delta(t, E) = \int_{-\infty}^{\infty} \frac{dp_0}{2\pi}\, e^{-ip_0 t} \Delta(p_0, \vec{p})$$

Same basic (bosonic) thermal operator

$$\begin{aligned}\Delta_{ab}^{(T)}(t, E) &= [1 + n(E)(1 - S(E)]\, \Delta_{ab}^{(0)}(t, E) \\ &= \mathcal{O}^{(T)}(E) \Delta_{ab}^{(0)}(t, E)\end{aligned}$$

- $\mathcal{O}^{(T)}(E)$ is *the same* for all propagator components $ab$
- Same proof as in the Euclidean formalism, but simpler:
- No need to extend the ranges of internal $t$-integrations
  ($-\infty < t < \infty$ in both cases)

# Extension to fermions

For a fermionic line:

$$\mathcal{O}_B^{(T)}(E) = 1 + n_{BE}(E)(1 - S(E)) \to \mathcal{O}_F^{(T)}(E) = 1 - n_{FD}(E)(1 - S(E))$$

(diagonal in spin space)

Fermionic propagator at finite temperature

$$S_{ab}^{(T)}(t, \vec{p}) = \mathcal{O}_F^{(T)}(E) S_{ab}^{(0)}(t, \vec{p})$$

Fermionic propagator at zero temperature

$$S_{++}^{(0)}(t, \vec{p}) = \frac{1}{2E}\left[\theta(t)A(E)e^{-i(E-i\varepsilon)t} + \theta(-t)B(E)e^{i(E-i\varepsilon)t}\right]$$
$$S_{+-}^{(0)}(t, \vec{p}) = \frac{1}{2E}B(E)e^{iEt}$$
$$S_{-+}^{(0)}(t, \vec{p}) = \frac{1}{2E}A(E)e^{-iEt}$$
$$S_{--}^{(0)}(t, \vec{p}) = \frac{1}{2E}\left[\theta(t)B(E)e^{i(E+i\varepsilon)t} + \theta(-t)A(E)e^{-i(E+i\varepsilon)t}\right]$$

where

$$A(E) = \gamma^0 E - \vec{\gamma} \cdot \vec{p} + m, \quad B(E) = -\gamma^0 E - \vec{\gamma} \cdot \vec{p} + m$$

# Non-zero chemical potential

Zero temperature (scalar case, Euclidean formalism):

$$\Delta^{(T=0,\mu)}(p_0,E) = \frac{1}{(p_0 - i\mu\,)^2 + E^2}$$

$$\Delta^{(T=0,\mu)}(\tau,E) = e^{\mu\tau}\Delta^{(T=0,\mu=0)}(\tau,E)$$

$$= \frac{1}{2E}\left[\theta(\tau)e^{-(E-\mu)\tau} + \theta(-\tau)e^{(E+\mu)\tau}\right]$$

Finite temperature and chemical potential

$$\Delta^{(T,\mu)}(\tau,E) = \frac{1}{2E}\left[\theta(\tau)\left\{(1+n_-)\,e^{-(E-\mu)\tau} + n_+e^{(E+\mu)\tau}\right\}\right.$$
$$\left. + \,\theta(-\tau)\left\{n_-e^{-(E-\mu)\tau} + (1+n_+)\,e^{(E+\mu)\tau}\right\}\right]$$

where

$$n_\pm = n(E \pm \mu)$$

# Non-zero chemical potential (continued)

Introduce a modified $T = 0, \mu = 0$ propagator (Inui+Kohyama+Niegawa, hep-ph/**0601**092),

$$\tilde{\Delta}^{(0,0)}(\tau, E, E_+, E_-) = \frac{1}{2E}\left[\theta(\tau)e^{-E_-\tau} + \theta(-\tau)e^{E_+\tau}\right]$$

Then:

$$\begin{aligned}\Delta^{(0,\mu)}(\tau, E) &= \tilde{\Delta}^{(0,0)}(\tau, E, E_+, E_-)\Big|_{E_\pm \to E \pm \mu} \\ &\equiv S(\mu)\tilde{\Delta}^{(0,0)}(\tau, E, E_+, E_-)\end{aligned}$$

and

$$\begin{aligned}\Delta^{(T,\mu)}(\tau, E) &= S(\mu)\left[1 + \hat{N}(1 - R(E))\right]\tilde{\Delta}^{(0,0)}(\tau, E, E_+, E_-) \\ &\equiv \mathcal{O}^{(T,\mu)}(E)\tilde{\Delta}^{(0,0)}(\tau, E, E_+, E_-)\end{aligned}$$

where

$$\begin{aligned}R(E)f(E, E_\pm) &= f(-E, -E_\mp) \\ \hat{N}f(E, E_\pm) &= n(E_\pm)f(E, E_\pm) \\ S(\mu)f(E, E_\pm) &= f(E, E \pm \mu)\end{aligned}$$

*Extension to fermions*: hep-th/**0601**224 and 227, PRD (2006)

# Outlook

## But is it useful?

For instance: retarded self-energy $\qquad \Pi_R(\omega,\mathbf{p}) = -\Pi_\beta(i(\omega+i\varepsilon),\mathbf{p})$

$\text{Im}\,\Pi_R(\omega,\mathbf{p}) \sim$ decay rate of particle propagating in the thermal medium

$$\text{Im}\,\gamma^{(T)}\,(i(\omega+i\varepsilon),E) = \hat{\mathcal{O}}^{(T)}(E)\,\text{Im}\,\gamma^{(0)}\,(i(\omega+i\varepsilon),E)$$

(reproduces Weldon's rules [1983])

Other "applications"? Please let me know!

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